The second type is

$$\int_0^1 \frac{\exp[-at^2 - bt^{-2}]}{(1 + t^2)^n} \qquad \text{(A1)}$$

To evaluate this integral, one starts from the case $n = 0$ considered in Ref 6; one thus has

$$\int_0^1 \exp[-at^2 - bt^{-2}]dt =$$

$$\frac{1}{4}\left(\frac{\pi}{a}\right)^{1/2} \{\exp 2(ab)^{1/2}\,\mathrm{erf}(a^{1/2} + b^{1/2}) +$$

$$\exp[-2(ab)^{1/2}]\,\mathrm{erf}(a^{1/2} - b^{1/2}) - \exp 2(ab)^{1/2} + \exp[-2(ab)^{1/2}]\}$$

Now, multiplying both sides of this equation by $\exp(-a)$ and integrating with respect to a, one has at the left-hand side the case $n = 1$ of Eq (A1) The integral at the right-hand side can be integrated by parts

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Unsteady Boundary-Layer Flow of a Non-Newtonian Fluid on a Flat Plate

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Nomenclature

a = constant of proportionality defined by Eq (1)
C_f = coefficient of friction
f = transformed stream function
L = characteristic length
n = non-Newtonian fluid index defined by Eq (1)
$\bar{U}$ = velocity at the edge of the boundary layer, divided by U_0
$\bar{u}$ = velocity in x direction, divided by U_0
$\bar{v}$ = velocity in y direction, divided by U_0
U_0 = reference velocity
$\bar{x}$ = coordinate along the surface, divided by L
$\bar{y}$ = coordinate normal to the surface divided by L
$\bar{t}$ = time, multiplied by U_0/L
R_{0n} = Reynolds number $\rho U_0^{2-n}L^n/a$
α_1,α_2 = constants
τ = shear stress
τ_0 = shear stress at the wall
ρ = density
η = similarity variable
ξ = function of x and t

In a recent report[1] in which an attempt was made to determine all possible similar solutions to the two-dimensional laminar boundary-layer equations for power-law non-Newtonian fluids, it was found that one of the unsteady flow solutions corresponds to the physical situation of impulsively started flow on a flat plate at zero angle of attack It is the purpose of this note to trace the development, by means of an interesting technique, of this particular solution from the general boundary-layer equations and similarity considerations In addition, numerical calculations are given for the case of impulsively started flow on an infinite flat plate, a simplification of the general problem

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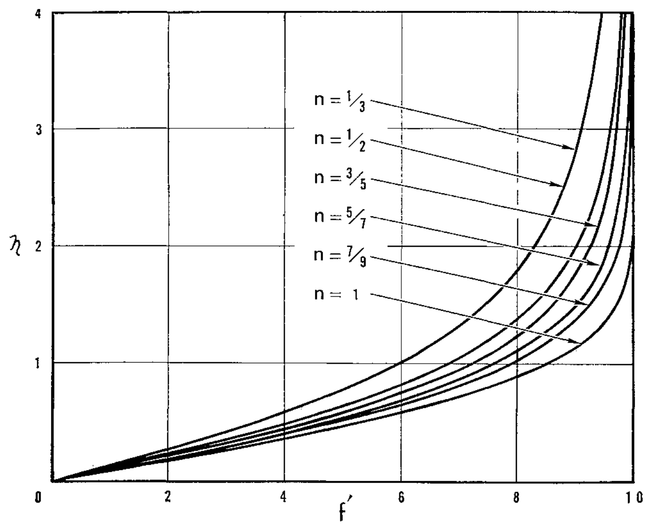

Fig 1 Similar boundary-layer profiles for impulsively started flow over an infinite flat plate for several non-Newtonian flow indices

The Ostwald-deWaele (power-law) model was chosen to describe the non-Newtonian fluid properties This is written, for the case of two-dimensional shear in the x direction due to a velocity gradient in the y direction, as

$$\tau_{yx} = a(du/dy)^n \qquad (1)$$

where a and n are constant properties of the fluid

The model described by Eq (1) is purely phenomenological; however, it is useful in that it approximately describes a great number of real non-Newtonian fluids It can be shown, however, that this model behaves properly under tensor transformation Use of this model alone assumes that the fluid is purely viscous, that is, no elasticity effects or anisotropic normal stresses are present Values of n less than one describe fluids which are shear-thinning, and values of n greater than one describe fluids which are sheaı-thickening (known less descriptively as pseudoplastic and dilatant, respectively) For n equal to one, the expression describes Newtonian fluids, and a equals the coefficient of viscosity

The purpose of the following description of the analysis contained in Ref 1 is to outline the method used in obtaining the ordinary differential equation, the boundary conditions, and the similarity parameter that corresponds to the physical situation of unsteady flow near a flat plate when the inviscid freestream is suddenly put into motion The technique is that of treating what Fenter terms the inverse problem—manipulating the general boundary-layer equations to find the conditions under which mathematical similarity (velocity profiles which transform linearly in y) exists [2]

The incompressible two-dimensional boundary-layer equations for power-law non-Newtonian fluids can be combined and transformed into a single ordinary differential equation in terms of a stream function $f' = \bar{u}/\bar{U}$ and a single independent variable η η is in general a function of $\bar{y}$ and $\xi(\bar{x},\bar{t})$, the form of which is dictated by similarity and the form of the differential equation The differential equation contains coefficients that must be constants in order to satisfy similarity With the assumption that the inviscid flow velocity is

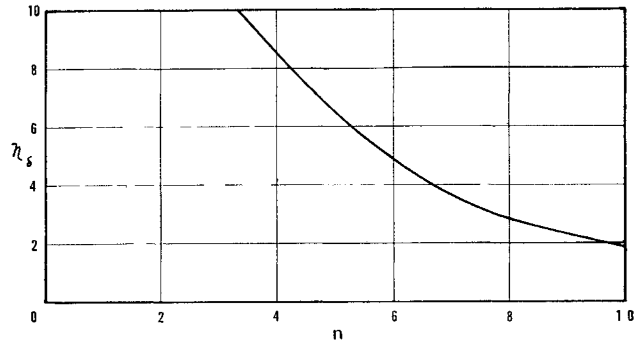

Fig 2 Similarity parameter for boundary-layer thickness as a function of the non-Newtonian fluid index

independent of $\bar{x}$ and $\bar{t}$, the expressions for the coefficients can be solved for $\xi(\bar{x},\bar{t})$ This gives an expression for η; and the conditions for similarity, imposed by the assumptions of the inviscid flow, can be written as an expression for the inviscid velocity, the differential equation, the boundary conditions, and the following similarity parameter:

$$\bar{U} = 1$$
$$f''' + 2\alpha_1 f(f'')^{2-} + 2\eta(f'')^{2-} = 0$$
$$f = f' = 0 \text{ at } \eta = 0 \tag{2}$$
$$f' \to 1 \text{ as } \eta \to \infty$$
$$\eta = \bar{y}\left[\frac{R_{0n}}{2n(n+1)[\alpha_1\bar{x} + \bar{t} + \alpha_2]}\right]^{1/(n+1)}$$

The physical flow described by Eqs (2) is that of impulsively started flow over a flat plate with a finite extent α_2 can be set equal to zero without loss of generality because it only serves to transform the origins of coordinates

If α_1 is set equal to zero, the convective term vanishes, and the physical problem is that of impulsively started flow over an infinite flat plate Equations (2) then can be written as

$$\bar{U} = 1$$
$$f''' + 2\eta(f'')^{2-n} = 0$$
$$f = f' = 0 \text{ at } \eta = 0 \tag{3}$$
$$f' \to 1 \text{ as } \eta \to \infty$$
$$\eta = \bar{y}\left[\frac{R_{0n}}{2n(n+1)(\bar{t})}\right]^{1/(\ +1)}$$

For $n \neq 1$, the ordinary differential equation in Eqs (3) can be integrated to give the following second-order equation:

$$f'' = [(n-1)(C_2 - \eta^2)]^{1/(n-1)} \tag{4}$$

where C_2 is a constant of integration

For $n = 1$ (Newtonian fluids), the differential equation in Eqs (3) has the solution

$$f' = \operatorname{erf}\eta \tag{5}$$

Equation (4) can be rewritten to give an expression for f', again for $n \neq 1$:

$$f' = \int_0^\eta [(n-1)(C_2 - \eta^2)]^{1/(n-1)} d\eta + C_3 \tag{6}$$

C_3 is shown to be zero by means of the first boundary condition Equation (6) is put into a more convenient form:

$$f' = \int_0^\eta \frac{d\eta}{(a + b\eta^2)^{m+1}} \tag{7}$$

where

$$a = C_2(n-1) \qquad b = 1 - n \qquad m = n/(1-n)$$

Equation (7) can be integrated in closed form for several values of n for $n < 1$ The second boundary condition permits the calculation of C_2 and $f''(0)$ where

$$f''(0) = [(n-1)C_2]^{1/(n-1)}$$

$f''(0)$ is related to the shear stress at the wall, and the dimensionless skin-friction coefficient can be written as

$$\frac{C_f}{[(t)^n R_0\ ]^{-1/(n+1)}} = \frac{2[f''(0)]^n}{[2n(n+1)]^{n/(n+1)}} \tag{8}$$

where $C_f \equiv 2\tau_0/\rho U^2$ Values needed to calculate the velocity profiles and the skin friction are collected in Table 1, and the velocity profiles are shown in Fig 1 The thickness of the boundary layer δ is defined in terms of η_δ, the value of η for which $f' = 0\ 995$:

$$\delta = \eta_\delta\left[\frac{2n(n+1)(\bar{t})}{R_{0n}}\right]^{1/(n+1)} \tag{9}$$

Values of η_δ also were calculated and are shown in Fig 2 as a function of n It is interesting to note from Eq (9) that δ is directly proportional to $(\bar{t})^{1/(n+1)}$ and inversely proportional to $R_{0n}{}^{1/(n+1)}$, as compared with the one-half power dependence for the particular case of Newtonian fluids

Table 1 Constant of integration and wall shear stress function for $n \leq 1$

n	C_2	$f''(0)$	$C_f[(t)^n R_0\ ]^{1/(n+1)}$
$\frac{1}{3}$	−1 838	0 737	1 861
$\frac{1}{2}$	−2 144	0 871	1 632
$\frac{3}{5}$	−2 563	0 940	1 509
$\frac{5}{7}$	−3 497	1 003	1 380
$\frac{7}{9}$	−4 468	1 088	1 369
1		1 128	1 128

For values of n greater than 1, Eq (6) does not satisfy the second boundary condition as stated However, if that boundary condition is changed to $f' = 1$ and $f'' = 0$ for $\eta \geq \eta$, where η is a finite boundary-layer thickness, then solutions may be found for $n > 1$ The physical significance of this branch of the problem is questioned, however No calculations have been made for this range of n

The significance of what has been presented is felt to lie in the technique used to obtain the conditions for similarity and in the generality permitted in presenting the numerical calculations It should be noted that the particular simple case of flow over an infinite plate chosen for the calculations can be derived initially as a second-order partial differential equation if the convective terms are neglected initially (see Refs 3 and 4) Of course, the more general case of unsteady flow over a flat plate with a finite leading edge, as given by Eqs (2), must be obtained through analysis of the complete boundary-layer equations as shown here

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